Technical Comments

Comment on "Optimal Measurement and Velocity Correction Programs for Midcourse Guidance"

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IN a paper by Denham and Spever,[1] the authors discuss the possibility of varying the times of midcourse velocity corrections to minimize a function ϕ, which involves the terminal deviation variances and a statistical measure of the total velocity change used for control

It is suggested that $\partial\phi/\partial t_j$ be calculated by changing t_j to $t_j + \Delta t_j$ and recalculating the $X(t)$ and $E(t)$ histories In such an approach Δt_j would have to be kept small to make the assumed linearization valid But when Δt is small, the computation of the derivative inevitably involves taking differences of nearly equal numbers, a procedure to be avoided if reasonable accuracy is to be attained The following procedure allows direct computation of the derivative

The basic problem is that of finding the partial derivatives

$$\frac{\partial X(t_i-)}{\partial t_k} \qquad \frac{\partial E(t_i-)}{\partial t_k} \qquad (k \leq i \leq n)$$

where t_k is the correction time to be varied and n is the number of time intervals t_n is the final time

We first derive the relations describing the propagation of perturbations in the variance matrices $\delta X(t)$ and $\delta E(t)$ in the periods between corrections From (3 7)† and (4 10), with $\Lambda = Q = 0$ (because the vehicle is in free-fall), and with $K = EH^TR^{-1}$ (for optimal state-variable estimation), we have

$$\dot{X} = XF^T + FX \tag{1}$$

$$\dot{E} = EF^T + FE - EH^T R^{-1} HE \tag{2}$$

Equation (1) is linear, and the corresponding perturbation equation, from (5 6), has the solution

$$\delta X_{i+1-} = \Phi_{i+1\ i}\delta X_{i+}\Phi_{i+1\ i}{}^T \qquad (i > k) \tag{3}$$

where, for brevity, we have omitted the arguments in favor of equivalent subscripts

Equation (2) is of the Riccati type and its perturbed solution (see Appendix A) satisfies the equation‡

$$\delta E_{i+1-} = [E_{i+1-}\Phi_{i+1\ i}{}^{-T}E_{i+}{}^{-1}]\delta E_{i+}[E_{i+}{}^{-1}\Phi_{i+1\ i}{}^{-1}E_{i+1-}] \qquad (i > k) \tag{4}$$

From Eq (5 11),

$$E_{i+} = E_{i-} + G_iQ_iG_i{}^T$$

where Q is the control implementation error covariance matrix It follows that

$$\delta E_{i+} = \delta E_{i-} + G_i\delta Q_iG_i{}^T$$

where δQ_i can be expressed in terms of δX_{i-} and δE_{i-} (if the control implementation error depends on the control) or may be zero Combining this with Eq (4), we can write δE_{iC+1-} in the form

$$\delta E_{i+1-} = \sum_j N_{Xij}\delta\Lambda_{i-}N_{Xij}{}' + \sum_j N_{Eij}\delta E_{i-}N_{Eij}{}' \tag{5}$$

The propagation of δX across a correction time is derived from Eq (5 7):

$$X_{i+} = X_{i-} + G_i[\Gamma(X - E)\Gamma^T + Q]_{i-}G_i{}^T + [(X - E)\Gamma^TG^T + G\Gamma(X - E)]_{i-}$$

where the matrix Γ (derived by Battin[3]) is the feedback matrix which determines the impulsive velocity correction $\Gamma\delta\hat{x}$ By perturbing Eq (5 7) we find

$$\delta X_{i+} = \delta X_{i-} + G_i\Gamma_i\delta X_{i-}\Gamma_i{}^TG_i{}^T + \delta X_{i-}\Gamma_i{}^TG_i{}^T + G_i\Gamma_i\delta X_{i-} - G_i\Gamma_i\delta E_{i-}\Gamma_i{}^TG_i{}^T - \delta E_{i-}\Gamma_i{}^TG_i{}^T - G_i\Gamma_i\delta E_{i-} + G_i\delta Q_iG_i{}^T \tag{6}$$

Combination of Eq (3) with Eq (6) gives an expression for δX_{i+1-} of the form

$$\delta X_{i+1-} = \sum_j M_{Xij}\delta X_{i-}M_{Xij}{}' - \sum_j M_{Eij}\delta E_{i-}M_{Eij}{}' \tag{7}$$

The forementioned recurrence formulas, Eqs (5) and (7), require, as initial conditions, the values of $\delta E(t_{k+1}-)$ and $\delta X(t_{k+1}-)$, where t_k is the correction time to be varied By differentiating Eq (5 7) with respect to time at t_k, we can express $\partial X(t_k+)/\partial t_k$ in terms of the time derivatives, at (t_k-), of the components of the right-hand side $\dot{X}(t_k-)$ is obtained from Eq (1), $\dot{G}$ from the original differential equations, $\dot{E}$ from Eq (2), $\dot{\Gamma}$ from Battin's derivation of Γ, and $\dot{Q}$ from the (time-varying) statistical properties of the control implementation error v If Q is also a function of X and E, its derivative must be calculated accordingly We can thus write $\delta X(t_k+)$ in the form

$$\delta X(t_k+) = M_{k+}\delta t_k \tag{8}$$

Now Eq (3) must be altered, for $i = k$, because $\Phi_{k+1\ k}$ changes as we vary t_k Hence,

$$\delta X_{k+1-} = \Phi_{k+1\ k}\delta X_{k+}\Phi_{k+1,k}{}^T + 2\Phi_{k+1\ k}X_{k+}\delta\Phi_{k+1\ k}{}^T \tag{9}$$

Now the transition matrix may be written as

$$\Phi_{k+1,k} = S(t_{k+1})S^{-1}(t_k) \tag{10}$$

where S is a nonsingular solution of

$$\dot{S}(t) = F(t)S(t)$$

Hence,

$$\frac{\partial}{\partial t_k}\Phi_{k+1\ k} = S(t_{k+1})\frac{d}{dt_k}S^{-1}(t_k) = -S_{k+1}S_k{}^{-1}\dot{S}_kS_k{}^{-1} = -\Phi_{k+1,k}F(t_k) \tag{11}$$

Hence, combining Eqs (8, 9, and 11), we may write $\delta X(t_{k+1}-)$ in the form

$$\delta X(t_{k+1}-) = M_{k+1}\delta t_k \tag{12}$$

In similar fashion, by differentiating (5 11), we can express $\delta E(t_k+)$ as

$$\delta E(t_k+) = N_{k+}\delta t_k \tag{13}$$

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† Equation numbers with decimals refer to the paper[1] under discussion

‡ The superscript $-T$ indicates the transposed inverse of a matrix

The relation between $\delta E(t_k+)$ and $\delta E(t_{k+1}-)$ is derived in Appendix B, and allows us to write

$$\delta E(t_{k+1}-) = N_{k+1}\delta t_k \quad (14)$$

With Eqs (12) and (14) and the recurrence formulas, Eqs (5) and (7), we can now evaluate the matrices M_i and N_i $(i > k + 1)$, which enable us to write

$$\delta X(t_i-) = M_i\delta t_k \qquad (i > k) \quad (15)$$

$$\delta E(t_i-) = N_i\delta t_k \qquad (i > k) \quad (16)$$

The matrices M_k and N_k are just the time derivatives of X and E at t_k, as determined from Eqs (1) and (2)

It should be noted that it is not necessary to solve additional differential equations in order to apply this procedure The matrices used in the recurrence formulas (5) and (7) are all quantities which presumably have already been computed for the nominal trajectory This is due principally to the fact that the vehicle follows an undisturbed free fall trajectory between corrections, which results in the uncoupling of Eq (A4) from Eq (A3) If Eq (A4) were not homogeneous, we would be forced to compute the transition matrix θ by integration of Eqs (A3) and (A4)

The analysis just given applies only if measurements are made continuously In the case of discrete measurements, Eq (2) would not be applicable, but would be replaced by Eq (7 1), and the remainder of the derivation would be altered accordingly

Appendix A: Perturbed Riccati Equation

The solution of Eq (2) is given by Kalman[2] as

$$E = [\theta_{21} + \theta_{22}E_0][\theta_{11} + \theta_{12}E_0]^{-1} \quad (A1)$$

E_0 and E here will represent values at the beginning and end of an interval between corrections, t_{i+} and t_{i+1-} The matrix

$$\theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \quad (A2)$$

is the transition matrix of

$$\dot{\mathbf{x}} = -F^T\mathbf{x} + H^TR^{-1}H\mathbf{w} \quad (A3)$$

$$\dot{\mathbf{w}} = F\mathbf{w} \quad (A4)$$

for the time interval in question

Obviously, $\theta_{21} = 0$ in this case, and $\theta_{22} = \Phi_{i+1\ i}$; θ_{11} is the transition matrix of the Eq (A3), and can be shown to be

$$\theta_{11} = \Phi_{i+1\ i}{}^{-T} \quad (A5)$$

From Eq (A1) we can deduce that, with $\theta_{21} = 0$,

$$\theta_{12} = E^{-1}\theta_{22} - \theta_{11}E_0{}^{-1} \quad (A6)$$

By perturbing Eq (A1) we find

$$\begin{aligned}\delta E &= [\theta_{22}\delta E_0][\theta_{11} + \theta_{12}E_0]^{-1} - \\ &\quad [\theta_{21} + \theta_{22}E_0][\theta_{11} + \theta_{12}E_0]^{-1}\theta_{12}\delta E_0[\theta_{11} + \theta_{12}E_0]^{-1} \\ &= [\theta_{22} - E\theta_{12}]\delta E_0[\theta_{11} + \theta_{12}E_0]^{-1}\end{aligned} \quad (A7)$$

With $\theta_{21} = 0$, this can be written, using Eq (A1),

$$\delta E = [\theta_{22} - E\theta_{12}]\delta E_0[\theta_{22}E_0]^{-1}E \quad (A8)$$

which, together with Eq (A6), gives

$$\delta E = [E\Phi^{-T}E_0{}^{-1}]\delta E_0[E_0{}^{-1}\Phi^{-1}E] \quad (A9)$$

Appendix B: Perturbed Riccati Equation with Variable Initial Time

We now consider the effect of varying the initial time on Eq (A9) We consider only the special case $\theta_{21} = 0$ E_0 and E represent E_{k+} and E_{k+1-} Equation (A1) can be written

$$E = \Phi E_0[\Phi^{-T} + \theta_{12}E_0]^{-1} \quad (B1)$$

and Eq (A6) can be written

$$\theta_{12} = E^{-1}\Phi - \Phi^{-T}E_0{}^{-1} \quad (B2)$$

Hence, from Eq (B1),

$$\begin{aligned}\delta E &= [\delta\Phi E_0 + \Phi\delta E_0]\ [\Phi^{-T} + \theta_{12}E_0]^{-1} - \\ &\quad E[\delta\Phi^{-T} + \delta\theta_{12}E_0 + \theta_{12}\delta E_0][\Phi^{-T} + \theta_{12}E_0]^{-1} \\ &= [\delta\Phi E_0 + \Phi\delta E_0]E_0{}^{-1}\Phi^{-1}E - \\ &\quad E[\delta\Phi^{-T} + \delta\theta_{12}E_0 + \theta_{12}\delta E_0]\ E_0{}^{-1}\Phi^{-1}E\end{aligned} \quad (B3)$$

where, using Eq (11),

$$\delta\Phi = -\Phi F_k\delta t_k \quad (B4)$$

$$\delta\Phi^{-T} = -\Phi^{-T}\delta\Phi^T\Phi^{-T} = \Phi^{-T}F_k{}^T\delta t_k \quad (B5)$$

The quantity $\delta\theta_{12}$ is determined by noting that the solution of Eqs (A3) and (A4) can be expressed as

$$\mathbf{w}(t) = \Phi(t,t_k)\mathbf{w}(t_k) \quad (B6)$$

$$\mathbf{x}(t) = \Phi^{-T}(t,t_k)\mathbf{x}(t_k) + \int_{t_k}^{t} \Phi^{-T}(t,\tau)H^T(\tau)R^{-1}(\tau)H(\tau)\mathbf{w}(\tau)d\tau \quad (B7)$$

Since the last term represents $\theta_{12}(t,t_k)\mathbf{w}(t_k)$, it follows that

$$\theta_{12}(t,t_k) = \int_{t_k}^{t} \Phi^{-T}(t,\tau)H^T(\tau)R^{-1}(\tau)H(\tau)\Phi(\tau,t_k)d\tau \quad (B8)$$

from which

$$\begin{aligned}\frac{\partial}{\partial t_k}\theta_{12}(t_{k+1},t_k) &= -\Phi^{-T}H_k{}^TR_k{}^{-1}H_k + \int_{t_k}^{t_{k+1}} \Phi^{-T}(t_{k+1},\tau) \times \\ &\quad H^T(\tau)R^{-1}(\tau)H(\tau)\frac{\partial}{\partial t_k}\Phi(\tau,t_k)d\tau \\ &= -\Phi^{-T}H_k{}^TR_k{}^{-1}H_k - \theta_{12}(t_{k+1},t_k)F_k\end{aligned} \quad (B9)$$

where we have made use of Eqs (B4) and (B8) in evaluating the last term

Combining Eqs (B2–B5 and B9), and simplifying,

$$\begin{aligned}\delta E &= E\Phi^{-T}[H_k{}^TR_k{}^{-1}H_k - F_k{}^TE_0{}^{-1} - E_0{}^{-1}F_k]\Phi^{-1}E\delta t_k + \\ &\quad [E\Phi^{-T}E_0{}^{-1}]\delta E_0[E_0{}^{-1}\Phi^{-1}E]\end{aligned} \quad (B10)$$

References

[1] Denham W F and Speyer J L, 'Optimal measurement and velocity correction programs for midcourse guidance, AIAA J 2, 896–907 (1964)

[2] Kalman R E and Bucy R S, 'New results in linear filtering and prediction theory, J Basic Eng Trans Am Soc Mech Engrs 83D, 95–108 (March 1961)

[3] Battin R H, A statistical optimizing navigation procedure for space flight, ARS J 32, 1681–1696 (1962)

Comment on "Design Analysis of Earth-Lunar Trajectories: Launch and Transfer Characteristics"

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THE subject paper[1] includes a most informative graphical presentation and interpretation of the launch and transfer characteristics of earth-lunar trajectories Declination

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